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CENTRAL LIMIT THEOREM FOR FOURIER TRANSFORMS OF STATIONARY PROCESSES

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We consider asymptotic behavior of Fourier transforms of stationary ergodic sequences with finite second moments. We establish a central limit theorem (CLT) for almost all frequencies and also an annealed CLT. The theorems hold for all regular sequences. Our results shed new light on the foundation of spectral analysis and on the asymptotic distribution of periodogram, and it provides a nice blend of harmonic analysis, theory of stationary processes and theory of martingales.

1. Introduction. In frequency or spectral domain analysis of time series, periodograms play a fundamental role. Since its introduction by Schuster (1898), periodograms have been used in almost all scientific fields. Given a realization $(X_j)_{j=1}^n$ of a stochastic process $(X_j)_{j\in\mathbb{Z}}$, the periodogram is defined as

$$I_n(\theta) = \frac{1}{2\pi n}\left|\sum_{j=1}^n X_j \exp(ji\theta)\right|^2, \qquad \theta \in \mathbb{R},$$

where $i = \sqrt{-1}$ is the imaginary unit. Periodogram is the building block in spectral domain analysis and a distributional theory is clearly needed in the related statistical inference. If (X_j) is a Gaussian process, then the Fourier transform

$$S_n(\theta) = \sum_{j=1}^n X_j \exp(ji\theta)$$

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is complex Gaussian. Fisher (1929) proposed a test for hidden periodicities and obtained a distributional theory based on i.i.d. Gaussian random variables. If (X_j) is not Gaussian, the distribution of $S_n(\theta)$ typically does not have a close form and one needs to resort to asymptotics. It is well known since Wiener and Wintner (1941) [see also Lacey and Terwilleger (2008)] that for any stationary sequence $(X_j)_{j\in\mathbb{Z}}$ in $\mathcal{L}^1$ (namely $E|X_0| < \infty$) there is a set Ω' of probability 1 such that for all θ and $\omega \in \Omega'$, $S_n(\theta)/n$ converges. Our problem is to investigate the speed of this convergence by providing a central limit theorem for the real and imaginary parts of $S_n(\theta)/\sqrt{n}$.

The above central limit problem was considered by many authors under various dependence conditions. We mention Rosenblatt [(1985), Theorem 5.3, page 131] who considered mixing processes; Brockwell and Davis [(1991), Theorem 10.3.2, page 347], Walker (1965) and Terrin and Hurvich (1994) discussed linear processes, and Wu (2005) treated mixingales. Other contributions can be found in Olshen (1967), Rootzén (1976), Yajima (1989), Woodroofe (1992), Walker (2000), Lahiri (2003) and Lin and Liu (2009) among others.

To establish an asymptotic theory for $S_n(\theta)/\sqrt{n}$, we shall provide the framework of stationary processes that can be introduced in several equivalent ways. We assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space. The marginal distribution is denoted by $\pi(A) = P(\xi_0 \in A)$. Next let $\mathcal{L}_0^2(\pi)$ be the set of functions such that $\int h^2\, d\pi < \infty$ and $\int h\, d\pi = 0$. Denote by $\mathcal{F}_k$ the σ-field generated by ξ_j with $j \le k$, $X_j = h(\xi_j)$. For any integrable random variable X we denote $E_k(X) = E(X|\mathcal{F}_k)$. We assume $h \in \mathcal{L}_0^2(\pi)$; in other words we assume $\|X_0\| := (E|X_0^2|)^{1/2} < \infty$ and $E(X_0) = 0$. Notice that any stationary sequence $(Y_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_j; j \le k)$ with the function $g(\xi_k) = Y_k$.

The stationary stochastic processes may be also introduced in the following alternative way. Let $T : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{F}$ satisfying $\mathcal{F}_1 \subseteq T^{-1}(\mathcal{F}_0)$. We then define the nondecreasing filtration $(\mathcal{F}_j)_{j\in\mathbb{Z}}$ by $\mathcal{F}_j = T^{-j}(\mathcal{F}_0)$ (referred to as the stationary filtration). Let X_0 be a random variable which is $\mathcal{F}_0$-measurable. We also define the stationary sequence $(X_j)_{j\in\mathbb{Z}}$ by $X_j = X_0 \circ T^j$. In this paper we shall use both frameworks.

The rest of the paper is structured as follows. The main results are presented in Section 2 and proved in Section 4. Our proofs in Section 4 provide an interesting blend of harmonic analysis, martingale approximation and theory of stationary processes. Examples of regular processes and further extensions are given in Section 3.

2. Main results. We shall assume that the following regularity condition holds:

$$E(X_0|\mathcal{F}_{-\infty}) = 0, \qquad P\text{-almost surely,} \tag{2.1}$$

and also that the sequence is stationary and ergodic. The regularity condition is quite mild and it is satisfied for many popular processes used in practice. Section 3 provides examples of stationary ergodic processes for which (2.1) holds.

We shall present first a central limit theorem for almost all frequencies. In Theorem 2.1, we let the parameter θ be in the space $[0, 2\pi]$, endowed with Borelian sigma algebra and Lebesgue measure λ. We denote by "$\Rightarrow$" the weak convergence, or convergence in distribution.

THEOREM 2.1. *Let $(X_k)_{k\in\mathbb{Z}}$ be a stationary ergodic process such that (2.1) is satisfied. Then for almost all $\theta \in (0, 2\pi)$, the following convergence holds:*

$$\lim_{n\to\infty} \frac{E|S_n(\theta)|^2}{n} = g(\theta) \qquad \text{(say)}, \tag{2.2}$$

where g is integrable over $\theta \in [0, 2\pi]$, and

$$\frac{1}{\sqrt{n}}[\mathrm{Re}(S_n(\theta)), \mathrm{Im}(S_n(\theta))] \Rightarrow [N_1(\theta), N_2(\theta)] \qquad \text{under } P, \tag{2.3}$$

where $N_1(\theta)$ and $N_2(\theta)$ are independent identically distributed normal random variables mean 0 and variance $g(\theta)/2$.

As implied by Lemma 4.2 in Section 4, $g(\theta)/(2\pi)$ is actually the spectral density associated with the spectral distribution function induced by the covariances

$$c_j = \mathrm{cov}(X_0, X_j), \qquad j \in \mathbb{Z}. \tag{2.4}$$

More specifically, by Herglotz's theorem [Brockwell and Davis (1991)], there exists a nondecreasing function G (the spectral distribution function) on $[0, 2\pi]$ such that, for all $j \in \mathbb{Z}$,

$$c_j = \int_0^{2\pi} \exp(ij\theta)\, dG(\theta). \tag{2.5}$$

Hence, by Lemma 4.2, G is absolutely continuous and the spectral density $G'(\theta)$ equals to $g(\theta)/(2\pi)$ almost surely. By (2.5) or (4.3), $\int_0^{2\pi} g(\theta)\, d\theta = 2\pi c_0$. So a nice implication of our results is that, under the regularity condition, we obtain an interesting representation of the spectral densities (see Lemma 4.2 for details).

Following the proof of Theorem 2.1, by the Cramér–Wold device, for

$$V_n(\omega,\theta) = \frac{S_n}{\sqrt{n}}(\omega,\theta),$$

we have that, for almost all pairs (θ',θ'') (Lebesgue), $V_n(\omega,\theta')$ and $V_n(\omega,\theta'')$ are asymptotically independent. In this sense Theorem 2.1 justifies the folklore in the spectral domain analysis of time series: the Fourier transforms of stationary processes are asymptotically independent Gaussian. Namely, in the spectral or Fourier domain, the Fourier-transformed processes are asymptotically independent, while the original process can be very strongly dependent (see Example 3.3).

Theorem 2.1 substantially improves the result in Wu (2005) that proves (2.3) under the following stronger condition:

$$\sum_{n=1}^{\infty} \frac{\|E(X_n|\mathcal{F}_0)\|^2}{n} < \infty. \tag{2.6}$$

We shall also establish the following characterization of the annealed CLT. Let Id_2 denote the identity 2×2 matrix.

THEOREM 2.2. *Under the same conditions as in Theorem 2.1 on the product space* $([0,2\pi]\times\Omega, \mathcal{B}\times\mathcal{F}, \lambda\times P)$ *we have*

$$\frac{1}{\sqrt{n}}[\mathrm{Re}(S_n(\theta)), \mathrm{Im}(S_n(\theta))] \Rightarrow [g(U)/2]^{1/2} N(0,\mathrm{Id}_2) \qquad \textit{under } \lambda\times P. \tag{2.7}$$

Here U *is a random variable independent of* $N(0,\mathrm{Id}_2)$ *and uniformly distributed on* $[0,2\pi]$ *and* $g(\cdot)$ *is defined by (2.2).*

Two types of stochastic processes can be considered concerning the partial sum $S_n(\theta)$. The process $V_n(\omega,\theta)$ indexed by θ is asymptotically Gaussian white noise. For another version, we consider

$$W_n(t,\omega,\theta) = \frac{S_{\lfloor nt\rfloor}}{\sqrt{n}}(\omega,\theta), \qquad 0\le t\le 1,$$

where $\lfloor x\rfloor = \max\{k\in\mathbb{Z} : k\le x\}$ is the integer part of x. We shall prove the following invariance principle:

PROPOSITION 2.1. *Assume that* (X_k) *is stationary ergodic and satisfies (2.1). Then* $W_n(t,\omega,\theta)$ *is tight in* $D(0,1)$*, and*

$$\begin{aligned}&[\mathrm{Re}(W_n(t,\omega,\theta)), \mathrm{Im}(W_n(t,\omega,\theta))]\\ &\qquad \Rightarrow [g(U)/2]^{1/2}[W'(t), W''(t)] \qquad \textit{under } \lambda\times P,\end{aligned}$$

where $(W'(t), W''(t))$ *are two independent standard Brownian motions independent of* U*, and* U *is a random variable uniformly distributed on* $[0,2\pi]$.

We now give some remarks and discussions.

Remark 2.1 (Nonadapted case). Our CLT also holds if X_0 is not $\mathcal{F}_0$-measurable. Then clearly the regularity condition we shall impose is

$$X_0 \text{ is } \mathcal{F}_\infty\text{-measurable} \quad \text{and} \quad E(X_0|\mathcal{F}_{-\infty}) = 0 \qquad \text{almost surely.}$$

Remark 2.2 (Adapted nonregular case). For general adapted sequences our CLT result still holds under centering. Let $\tilde{S}_n(\theta) = S_n(\theta) - E(S_n(\theta)|\mathcal{F}_{-\infty})$ and $\tilde{X}_k = X_k - E(X_k|\mathcal{F}_{-\infty})$, where $E(X_k|\mathcal{F}_{-\infty})$ denotes the following limit that holds almost surely and in $\mathcal{L}^2$

$$\lim_{n\to\infty} E(X_k|\mathcal{F}_{-n}) = E(X_k|\mathcal{F}_{-\infty}).$$

Then, for almost all $\theta \in (0, 2\pi)$, we have

$$\frac{1}{\sqrt{n}}[\mathrm{Re}(\tilde{S}_n(\theta)), \mathrm{Im}(\tilde{S}_n(\theta))] \Rightarrow N\left(0, \frac{g(\theta)}{2}\,\mathrm{Id}_2\right) \qquad \text{under } P$$

by applying Theorem 2.1 to the stationary sequence $\tilde{X}_k$. Therefore the conclusion of Theorem 2.1 holds if we replace the assumption of regularity (2.1) by the following: for λ-almost all θ

$$\frac{1}{\sqrt{n}} E(S_n(\theta)|\mathcal{F}_{-\infty}) \to 0 \qquad \text{in probability.}$$

Now, since $\|E(S_n(\theta)|\mathcal{F}_{-\infty})\|_2 \le \|E(S_n(\theta)|\mathcal{F}_{-n})\|_2 \le \|E(S_n(\theta)|\mathcal{F}_0)\|_2$, Theorem 2.1 still holds if

$$\frac{1}{\sqrt{n}}\|E(S_n(\theta)|\mathcal{F}_{-n})\|_2 \to 0 \qquad \text{as } n \to \infty$$

or under the condition

$$\frac{1}{\sqrt{n}}\|E(S_n(\theta)|\mathcal{F}_0)\|_2 \to 0 \qquad \text{as } n \to \infty. \tag{2.8}$$

Remark 2.3 (Conditional CLT). Since we use in the proof martingale approximation actually our CLT is a conditional CLT, that allows for a random change of measure. See Hall and Heyde (1980) and Dedecker and Merlevède (2002).

Remark 2.4 (Periodogram). We notice that, as a consequence of Theorem 2.1, for sequences satisfying (2.1) the periodogram $n^{-1}|S_n(\theta)|^2$ is asymptotically distributed as $\frac{g(\theta)}{2}\chi^2(2)$ for almost all frequencies.

REMARK 2.5 (Resulting identities). By using the invariance principle in Proposition 2.1 we can get the convergence of many interesting functionals of $S_n(\theta)$ and periodograms. As a consequence of Proposition 2.1 we can get, for instance,

$$\frac{1}{\|X_0\|^2 n\pi}\int_0^{2\pi} E\left[\max_{1\le m\le n}\sum_{k=1}^{m} X_k\cos(k\theta)\right]^2 d\theta$$
$$\to E\left|\sup_{0\le t\le 1} W(t)\right|^2 = \int_0^\infty 2[1-\Phi(\sqrt{y})]\,dy = 1$$

by noting that $P(\sup_{0\le t\le 1} W(t)\ge u) = 2P(W(1)\ge u)$ for $u\ge 0$. Here $\Phi(\cdot)$ is the standard Gaussian distribution function.

3. Examples. Here we present several examples of processes for which the conclusions of Theorems 2.1 and 2.2 hold.

Clearly condition (2.1) is satisfied if the left tail sigma field $\mathcal{F}_{-\infty}$ is trivial. These processes are called regular [see Chapter 2, Volume 1 in Bradley (2007)]. Notice, however, that our condition (2.1) refers rather to the function $X_0 = f(\xi_0)$ in relation to the tail field $\mathcal{F}_\infty$.

EXAMPLE 3.1 (Mixing sequences). We shall introduce the following mixing coefficients: for any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ define the strong mixing coefficient

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{|P(A\cap B) - P(A)P(B)| : A\in\mathcal{A}, B\in\mathcal{B}\}$$

and the ρ-mixing coefficient, also known as maximal coefficient of correlation

$$\rho(\mathcal{A},\mathcal{B}) = \sup\{\mathrm{Cov}(X,Y)/\|X\|_2\|Y\|_2 : X\in\mathcal{L}^2(\mathcal{A}), Y\in\mathcal{L}^2(\mathcal{B})\}.$$

For the stationary sequence of random variables $(X_k)_{k\in\mathbb{Z}}$, $\mathcal{F}^n$ denotes the σ-field generated by X_i with indices $i\ge n$, and $\mathcal{F}_m$ denotes the σ-field generated by X_i with indices $i\le m$. The sequences of coefficients $\alpha(n)$ and $\rho(n)$ are then defined by

$$\alpha(n) = \alpha(\mathcal{F}_0,\mathcal{F}^n) \quad \text{and} \quad \rho(n) = \rho(\mathcal{F}_0,\mathcal{F}^n),$$

respectively. For strongly mixing sequences, namely the strong mixing coefficients $\alpha(n)\to 0$, the tail sigma field is trivial [see Claim 2.17a in Bradley (2007)]. Examples of this type include Harris recurrent Markov chains. If $\lim_{n\to\infty}\rho(n) < 1$, then the tail sigma field is also trivial [see Proposition 5.6 in Bradley (2007)].

EXAMPLE 3.2 (Functions of Gaussian processes). Assume (Y_k) is a stationary Gaussian sequence and define $X_n = f(Y_k, k\le n)$. Let f be such that

$E(X_0)=0$ and $E(X_0^2)<\infty$. Since any Gaussian sequence can be represented as a function of i.i.d. random variables, the process is then regular. Rosenblatt (1981) considered Fourier transforms of functionals of Gaussian sequences.

EXAMPLE 3.3 (Functions of i.i.d. random variables). Let ε_k be i.i.d. and consider $X_n = f(\varepsilon_k, k \le n)$. These are regular processes and therefore Theorems 2.1 and 2.2 are applicable. Examples include linear processes, functions of linear processes and iterated random functions [Wu and Woodroofe (2000), among others]. For example, let $X_n = \sum_{j=0}^{\infty} a_j \varepsilon_{n-j}$, where ε_j are i.i.d. with mean 0 and variance 1, and a_j are real coefficients with $\sum_{j=1}^{\infty} a_j^2 < \infty$. In this case X_n is well defined, and, by Lemmas 4.1 and 4.2, the spectral density is $g(\theta)/(2\pi)$, where

$$g(\theta) = \left| \sum_{j=0}^{\infty} a_j \exp(ij\theta) \right|^2 .$$

As a specific example, let $a_j = j^{-1/2}/\log j$, $j \ge 2$, and $a_0 = a_1 = 1$. By elementary manipulations, the covariances $c_j \sim (\log j)^{-1}$, which decays very slowly as $j \to \infty$, hence suggesting strong dependence. For this example, condition (2.6) is violated. By the Tauberian theorem, as $\theta \to 0$, $g(\theta) \sim \pi/(|\theta| \log^2 |\theta|)$, which has a pole at $\theta = 0$.

EXAMPLE 3.4 (Reversible Markov chains). As before, let ξ_j be a stationary ergodic Markov chain with values in a measurable space. We use the notation and constructions from the Introduction. The marginal distribution and the transition metric are denoted by $\pi(A) = P(\xi_0 \in A)$ and $Q(\xi_0, A) = P(\xi_1 \in A|\xi_0)$. In addition Q denotes the operator $Qf(\xi) = \int f(z) Q(\xi, dz)$. Let Q^* be the adjoint operator of the restriction of Q to $\mathcal{L}^2(\pi)$ and assume $Q = Q^*$. Then, for any $f \in \mathcal{L}_0^2(\pi)$ the central limit theorem of Theorem 2.1 holds. To see this we shall verify condition (2.8). By spectral calculus

$$\|E(S_n(\theta)|\mathcal{F}_0)\|^2 = \int_{-1}^{1} \left| \sum_{k=1}^{n} (t \exp(i\theta))^k \right|^2 \rho_f(dt),$$

where ρ_f denotes the spectral measure of f with respect to Q [see, e.g., Borodin and Ibragimov (1994) for this identity]. For $\theta \ne 0$, π and $-1 \le t \le 1$, we have

$$\begin{aligned}\left| \sum_{k=1}^{n} (t \exp(i\theta))^k \right|^2 &\le 4|1 - \exp(i\theta)t|^{-2} \\ &= 4(1 + t^2 - 2t\cos\theta)^{-1} \le 4(1 - (\cos\theta)^2)^{-1}.\end{aligned}$$

Therefore, for λ-almost all θ

$$\frac{1}{n}\|E(S_n(\theta)|\mathcal{F}_0)\|^2 \to 0.$$

4. Proofs. We shall establish first some preparatory lemmas. The almost sure convergence in Lemma 4.1 was shown by Wu (2005). The convergence in $\mathcal{L}^2$ is new here. For $k \in \mathbb{Z}$ we define the projection operator by

$$\mathcal{P}_k \cdot = E(\cdot|\mathcal{F}_k) - E(\cdot|\mathcal{F}_{k-1}). \tag{4.1}$$

LEMMA 4.1. *Let*

$$T_n(\theta) = \sum_{j=0}^{n} X_j \exp(ji\theta) = X_0 + S_n(\theta).$$

Under (2.1), for λ-almost all θ (Lebesgue), we have

$$\mathcal{P}_0 T_n(\theta) \to \sum_{l=0}^{\infty} \mathcal{P}_0 X_l \exp(li\theta) =: D_0(\theta), \qquad P\text{-almost surely and in } \mathcal{L}^2. \tag{4.2}$$

PROOF. By (2.1), $\sum_{k\in\mathbb{Z}} \|\mathcal{P}_k X_0\|_2^2 = \|X_0\|_2^2 < \infty$, we have $\sum_{k\in\mathbb{Z}} |\mathcal{P}_0 X_k|^2 < \infty$, P-almost surely. Therefore by Carleson's (1966) theorem, for almost all ω, $\sum_{1\le k\le n} (\mathcal{P}_0 X_k)\exp(ik\theta)$ converges λ-almost surely, where λ is the Lebesgue measure on $[0, 2\pi]$. Denote the limit by $D_0 = D_0(\theta)$. We now consider the set

$$A = \{(\theta,\omega) \subset [0,2\pi] \times \Omega, \text{ where } \{\mathcal{P}_0 S_n(\theta)\}_n \text{ does not converge}\}$$

and notice that almost all sections for ω fixed have Lebesgue measure 0. So by Fubini's theorem the set A has measure 0 in the product space and therefore, again by Fubini's theorem, almost all sections for θ fixed have probability 0. It follows that for almost all θ, $\mathcal{P}_0(S_n(\theta)) \to D_0$ almost surely under P. Next, by the maximal inequality in Hunt and Young (1974), there is a constant C such that

$$\int_0^{2\pi} \Bigl[\sup_n |\mathcal{P}_0(S_n(\theta))|^2\Bigr] \lambda(d\theta) \le C \sum_k |\mathcal{P}_0 X_k|_2^2,$$

and then we integrate

$$\int_0^{2\pi} E\Bigl[\sup_n |\mathcal{P}_0(S_n(\theta))|^2\Bigr] \lambda(d\theta) \le C\|X_0\|_2^2 < \infty.$$

Therefore,

$$E\Bigl[\sup_n |\mathcal{P}_0(S_n(\theta))|^2\Bigr] < \infty \qquad \text{for almost all } \theta.$$

Since $|\mathcal{P}_0 S_n(\theta)| < \sup_n |\mathcal{P}_0 S_n(\theta)|$, and the last one is integrable for almost all θ, by the Lebesgue dominated convergence we have that $\mathcal{P}_0(S_n(\theta))$ converges in $\mathcal{L}^2$. $\square$

LEMMA 4.2. *Let $g(\theta) = E|D_0(\theta)|^2$. For all $j \in \mathbb{Z}$, we have*

$$\int_0^{2\pi} g(\theta) \exp(ji\theta)\, d\theta = 2\pi c_j, \tag{4.3}$$

where c_j are defined by (2.4). So (c_j) are the Fourier coefficients of g. Additionally, for almost all θ,

$$\lim_{n\to\infty} \frac{E|S_n(\theta)|^2}{n} = g(\theta). \tag{4.4}$$

PROOF. Without loss of generality we let $j \ge 0$ and $j < n$. As before, let

$$\mathcal{P}_0 T_n(\theta) = \sum_{l=0}^{n} \mathcal{P}_0 X_l \exp(li\theta).$$

By elementary trigonometric identities, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |\mathcal{P}_0 T_n(\theta)|^2 \exp(ji\theta)\, d\theta = \sum_{l=0}^{n-j} (\mathcal{P}_0 X_l)(\mathcal{P}_0 X_{l+j}). \tag{4.5}$$

Since $X_j = \sum_{l\in\mathbb{Z}} \mathcal{P}_l X_j$, by orthogonality of martingale differences and stationarity we have that

$$\begin{aligned} c_j &= \lim_{N\to\infty} E\left[\left(\sum_{l=-N}^{0} \mathcal{P}_l X_0\right)\left(\sum_{l=-N}^{0} \mathcal{P}_l X_j\right)\right] \\ &= \lim_{N\to\infty} \sum_{l=-N}^{0} E[(\mathcal{P}_l X_0)(\mathcal{P}_l X_j)] \\ &= \lim_{n\to\infty} \sum_{l=0}^{n-j} E[(\mathcal{P}_0 X_l)(\mathcal{P}_0 X_{l+j})]. \end{aligned}$$

By (4.5) and the Lebesgue dominated convergence theorem, as in the proof of Lemma 4.1, (4.3) follows in view of Hunt's maximal inequality since $\sup_n |\mathcal{P}_0 \times T_n(\theta)|$ is integrable.

Now we prove (4.4). By stationarity, we have

$$\frac{1}{n} E|S_n(\theta)|^2 = \frac{1}{n} \sum_{j=1}^{n} \sum_{l=1}^{n} E(X_j X_l) \exp(ij\theta) \exp(-il\theta)$$

$$= \frac{1}{n}\sum_{j=1}^{n-1}\sum_{l=1}^{n} c_{j-l}\exp((j-l)i\theta) \tag{4.6}$$

$$= \sum_{j=-(n-1)}^{n-1}\left(1-\frac{|j|}{n}\right)c_j\exp(ij\theta).$$

Namely $E|S_n(\theta)|^2/n$ is the Cesaro average of the sum $\sum_{j=-l}^{l} c_j\exp(ij\theta)$. Note that $g(\theta)=\|D_0(\theta)\|^2$ is integrable over $[0,2\pi]$. Therefore by the Fejér–Lebesgue theorem [cf. Bary (1964), page 139 or Theorem 15.7 in Champeney (1989)], (4.4) holds for λ-almost all $\theta\in[0,2\pi]$ (Lebesgue). $\square$

REMARK 4.1. In the proof of Lemma 4.2, (4.6) implies that the sequence $(c_j\exp(ij\theta))$ is Cesaro summable. It turns out that, generally speaking, $\sum_{j=0}^{\infty} c_j\times\exp(ij\theta)$ may not exist for almost all θ. Consider the example in Kolmogorov (1923) [see also Theorem 3.1, page 305, in Zygmund (2002)]: there exists a sequence of nonnegative trigonometric polynomials f_n with constant term $1/2$, a sequence of positive integers $q_k\to\infty$ and a positive sequence $A_n\to\infty$, such that the function

$$g(x)=\sum_{k=1}^{\infty}\frac{f_{n_k}(q_k x)}{A_{n_k}^{1/2}}$$

is integrable. However, for almost all θ, the Fourier sum $\sum_{l=1}^{\infty} c_l\exp(li\theta)$ diverges, where $c_l=\int_0^{2\pi} g(\theta)\exp(li\theta)\,d\theta$ is the Fourier coefficient of g. Let $G(x)=\int_0^x g(u)\,du$. By Herglotz's theorem [Brockwell and Davis (1991)], there exists a stationary process (X_j) such that its spectral distribution function is G and its covariance function is c_l.

LEMMA 4.3. *Assume (2.1). On the product space* $([0,2\pi]\times\Omega,\mathcal{B}\times\mathcal{F},\lambda\times P)$ *we have that*

$$\left(\frac{\max_{1\le k\le n}|S_k(\theta)|^2}{n}\right)_{n\ge 1} \qquad \textit{is uniformly integrable.}$$

PROOF. Let m be a positive integer. We shall decompose the partial sums in a sum of m martingales and a remainder in the following way:

$$\frac{S_k(\theta)}{\sqrt{n}}=\frac{1}{\sqrt{n}}\sum_{l=1}^{k}\exp(il\theta)\sum_{j=0}^{m-1}\mathcal{P}_{l-j}(X_l)+\frac{1}{\sqrt{n}}\sum_{l=1}^{k}\exp(il\theta)E_{l-m}(X_l).$$

Notice that for any $0\le j\le m-1$,

$$\sum_{l=1}^{k}\exp(il\theta)\mathcal{P}_{l-j}(X_l)$$

is a martingale adapted to the filtration $(\mathcal{B}\times\mathcal{F}_k)$. Moreover, since $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence with variables square integrable, it follows that $(X_k^2)_{k\in\mathbb{Z}}$ is a uniformly integrable sequence. This fact implies that for j fixed the sequence $(\mathcal{P}_{k-j}(X_k))_{k\in\mathbb{Z}}$ is also uniformly integrable. It follows that $\sum_{l=1}^k \exp(il\theta)\times\mathcal{P}_{l-j}(X_l)$ is a martingale with uniformly integrable differences under the measure $\lambda\times P$. It is known that for a martingale with uniformly integrable differences we have

$$\frac{1}{n}\max_{1\le k\le n}\left|\sum_{l=1}^{k}\exp(il\theta)\mathcal{P}_{l-j}(X_l)\right|^2$$

is uniformly integrable [see, e.g., Dedecker and Rio (2000), Proposition 1].

The result follows since by Hunt and Young (1974) maximal inequality

$$\int_0^{2\pi}\max_{1\le k\le n}\left|\sum_{l=1}^{k}\exp(il\theta)E_{l-m}(X_l)\right|^2 d\theta\le C\sum_{l=1}^{n}|E_{l-m}(X_l)|^2,$$

and therefore, denoting by $\mathbf{E}$ the expected value with respect to $\lambda\times P$, we have by regularity condition (2.1) that

$$\begin{aligned}\frac{1}{n}\mathbf{E}\left[\max_{1\le k\le n}\left|\sum_{l=1}^{k}\exp(il\theta)E_{l-m}(X_l)\right|^2\right]&\le\frac{C}{n}\sum_{l=1}^{n}E|E_{l-m}(X_l)|^2\\&=CE|E_{-m}(X_0)|^2\to 0\end{aligned}$$

as $m\to\infty$ uniformly in n. So, the uniform integrability follows. □

4.1. *Proof of Theorem 2.1.* The first assertion of Theorem 2.1 is just Lemma 4.1. We now prove (2.3).

Step 1. *The construction of martingale.*

Define the projector operator by (4.1). Then we construct as in Lemma 4.1

$$\mathcal{P}_1(S_n(\theta))=E(S_n(\theta)|\mathcal{F}_1)-E(S_n(\theta)|\mathcal{F}_0)=\sum_{k=1}^{n}\exp(ik\theta)\mathcal{P}_1(X_k)$$

and then by Lemma 4.1 for almost all θ

$$\mathcal{P}_1(S_n(\theta))\to D_1(\theta)\qquad\text{in }\mathcal{L}^2.$$

To verify it is a martingale we start from

$$E(\mathcal{P}_1(S_n(\theta))|\mathcal{F}_0)=0\qquad\text{almost surely under }P$$

and by the contractive property of the conditional expectation

$$0=E(\mathcal{P}_1(S_n(\theta))|\mathcal{F}_0)\to E(D_1(\theta)|\mathcal{F}_0)\qquad\text{in }\mathcal{L}^2.$$

We then construct the sequence of stationary martingale differences $(D_k(\theta))_{k\geq 1}$, given by

$$\mathcal{P}_k(S_{n+k}(\theta)-S_k(\theta))\to \exp(ik\theta)D_k(\theta) \qquad \text{in } \mathcal{L}^2.$$

Step 2. *Martingale approximation.*

Denote by

$$M_n(\theta)=\sum_{1\leq k\leq n}\exp(ik\theta)D_k(\theta).$$

We show that, for almost all θ,

$$\frac{E|S_n(\theta)-M_n(\theta)|^2}{n}\to 0. \tag{4.7}$$

To this end, note that $S_n(\theta)-E(S_n(\theta)|\mathcal{F}_0)$ and $E(S_n(\theta)|\mathcal{F}_0)$ are orthogonal, we have

$$\|S_n(\theta)\|^2=\|S_n(\theta)-E(S_n(\theta)|\mathcal{F}_0)\|^2+\|E(S_n(\theta)|\mathcal{F}_0)\|^2. \tag{4.8}$$

For those θ such that (4.2) holds, we have, by the orthogonality of martingale differences and the stationarity, that

$$\begin{aligned}
&\|S_n(\theta)-E(S_n(\theta)|\mathcal{F}_0)-M_n(\theta)\|^2\\
&\quad=\sum_{k=1}^{n}\|\mathcal{P}_k(S_n(\theta)-M_n(\theta))\|^2=\sum_{k=1}^{n}\|\mathcal{P}_kS_n(\theta)-e^{ik\theta}D_k(\theta)\|^2\\
&\quad=\sum_{k=1}^{n}\|\mathcal{P}_0T_{n-k}(\theta)-D_0(\theta)\|^2=o(n).
\end{aligned} \tag{4.9}$$

Hence, by (4.8) and (4.4), we have

$$\limsup_{n\to\infty}\frac{\|E(S_n(\theta)|\mathcal{F}_0)\|^2}{n}=\limsup_{n\to\infty}\frac{\|S_n(\theta)\|^2-\|M_n(\theta)\|^2}{n}=0$$

by noting that $\|M_n(\theta)\|^2=n\|D_0(\theta)\|^2$. Hence we have (4.7) in view of (4.9).

Step 3. *The CLT for the approximating martingale.*

It remains just to prove central limit theorem for complex valued martingale

$$\frac{1}{\sqrt{n}}\sum_{1\leq k\leq n}\exp(ik\theta)D_k(\theta).$$

As a matter of fact we shall provide a central limit theorem for the real part and imaginary part and show that in the limit they are independent. The

proof was carefully written down in Wu (2005). By the Cramér–Wold device we have to study the limiting distribution of the martingale

$$\frac{1}{\sqrt{n}} \sum_{1\le k\le n} [s\,\mathrm{Re}(\exp(ik\theta)D_k(\theta)) + t\,\mathrm{Im}(\exp(ik\theta)D_k(\theta))].$$

By the Raikov-type of argument, in order to prove the CLT we have only to show

$$\frac{1}{n} \sum_{1\le k\le n} [s\,\mathrm{Re}(\exp(ik\theta)D_k(\theta)) + t\,\mathrm{Im}(\exp(ik\theta)D_k(\theta))]^2 \to^p \frac{(s^2+t^2)\sigma^2(\theta)}{2}.$$

This follows from combining the following two facts. First by stationarity

$$\frac{1}{n} \sum_{1\le k\le n} |\exp(ik\theta)D_k(\theta)|^2 = \frac{1}{n} \sum_{1\le k\le n} |D_k(\theta)|^2 \to E|D_0(\theta)|^2$$

and then by Lemma 5 in Wu (2005) for almost all θ

$$\frac{1}{n} \sum_{1\le k\le n} [\exp(ik\theta)D_k(\theta)]^2 \to 0.$$

The rest is simple algebra.

4.2. *Proof of Theorem 2.2.* Consider the product space $([0,2\pi]\times\Omega, \mathcal{B}\times\mathcal{F}, \lambda\times P)$. Let $\mathbf{P}=\lambda\times P$ and $\mathbf{E}$ the corresponding expected value. We already have shown in Theorem 2.1, that for λ-almost all θ,

$$E\exp\left[\frac{i}{\sqrt{n}}(s\,\mathrm{Re}(S_n(\theta)) + t\,\mathrm{Im}(S_n(\theta)))\right] \to \exp\left[-\frac{(s^2+t^2)g(\theta)}{4}\right]$$

for $s,t\in\mathbb{R}$. Then we integrate with θ and by the dominated convergence theorem we obtain

$$\mathbf{E}\exp\left[\frac{i}{\sqrt{n}}(s\,\mathrm{Re}(S_n(\theta)) + t\,\mathrm{Im}(S_n(\theta)))\right] \to \frac{1}{2\pi}\int_0^{2\pi} \exp\left[-\frac{(s^2+t^2)g(\theta)}{4}\right] d\theta.$$

We then identify the limiting distribution as being a mixture of two independent random variables: a standard normal variable with a variable uniformly distributed on $[0,2\pi]$.

4.3. *Proof of Proposition 2.1.* It is easy to see that the finite-dimensional distributions are convergent. So we just have to prove tightness. From Billingsley (1999), stationarity and standard considerations this follows by Lemma 4.3.

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